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Abstract

Bayes estimation of the reliability function for the inverse Gaussian distribution is discussed. For the case that the mean lifetime is known, Bayes estimators are obtained with Jeffreys' noninformative prior and with the natural conjugate prior for the scale parameter. In the case that both parameters are unknown, an estimator of reliability is suggested which is based on the Bayes estimator obtained for the case that the mean lifetime is known. This estimator is not Bayes but compares favorably with the maximum likelihood and minimum variance unbiased estimators as indicated by computer simulations.

Key Words: Reliability function; Life testing.

1. INTRODUCTION

The two-parameter inverse Gaussian distribution with probability density function (pdf) in the form

$$f(x|\mu,\lambda) = (\lambda/2\pi x^3)^{\frac{1}{2}} \exp\,[-\lambda(x-\mu)^2/2\mu^2 x],\ x > 0,\ \mu > 0,\ \lambda > 0, \qquad (1.1)$$

has been studied in the reliability or life testing context by several authors [6, 9, 10]. The parameters in (1.1) have more appealing physical interpretations for life testing situations than other parametric forms of the pdf. The mean life for the lifetime model (1.1) is μ, and λ is a shape parameter. The variance is μ^3/λ so μ is not a location parameter in the usual sense. Chhikara and Folks [6] state some advantages of using the inverse Gaussian distribution as a lifetime model over the log-normal distribution, and the wide variety of shapes generated by the pdf (1.1) makes it a competitor to other lifetime distributions. In addition, the inverse Gaussian distribution arises as the first passage time distribution of a Brownian motion process [7], justifying its use as a duration time or lifetime model on a physical basis. Several results have also been obtained concerning tests for drift in Brownian motion processes (for example [5, 9, 13]).

Tweedie [15, 16] and Chhikara and Folks [3, 4, 6], among others, have studied various sampling theory inferences concerning (1.1). Estimation for a three-parameter inverse Gaussian distribution was investigated recently by Padgett and Wei [12].

The cumulative distribution function (cdf) of (1.1) has been obtained in closed form by Shuster [14] and Chhikara and Folks [6], and the survival function or reliability is given in the form

$$R(t|\mu,\lambda) = \Phi[(\lambda/t)^{\frac{1}{2}}(1-t/\mu)] - \exp(2\lambda/\mu)\ \Phi[-(\lambda/t)^{\frac{1}{2}}(1+t/\mu)],\ t > 0, \qquad (1.2)$$

where Φ denotes the cdf of the standard normal distribution. The minimum variance unbiased estimator of $R(t|\mu,\lambda)$ was derived by Chhikara and Folks [6] and lower confidence bounds for (1.2) were given in [10]. However, due to the complicated nature of the expression (1.2), other inferences concerning the reliability seem to be difficult.

Recently, Banerjee and Bhattacharyya [1] presented a Bayesian analysis of the inverse Gaussian distribution in a different parametric form than (1.1) (see Johnson and Kotz [8] for other forms of (1.1)). As was stated in [1], Bayesian inferences concerning reliability are extremely difficult and require numerical integration to plot the posterior pdf or to determine HPD intervals for reliability. However, Bayes estimation can be performed in come cases, and it is the purpose of this note to present Bayes estimators of $R(t|\mu,\lambda)$ for the case that the mean life μ is known, which is reasonable in many reliability problems. Vague priors as well as a conjugate family of prior distributions are used. In the case that μ and λ are both unknown, an estimator of (1.2) is propósed which, as indicated by some Monte Carlo simulation results, is overall as good as the minimum variance unbiased estimator or maximum likelihood estimator given in [6] and is simpler to calculate than the minimum variance unbiased estimator. The results bear a remarkable similarity to those for the two-parameter log-normal model given by Padgett and Wei [11].

2. ESTIMATION OF RELIABILITY

For a random sample $\underline{x} = (x_1, \ldots, x_n)$ from the inverse Gaussian distribution (1.1), the likelihood function is given by

$$\ell(\lambda,\mu|\underline{x}) = \prod_{i=1}^{n} x_i^{-3/2} \exp\left[-\frac{\lambda}{2}\left(\frac{\bar{x}}{\mu^2} + \sum_{i=1}^{n} x_i^{-1}\right)\right] \tag{2.1}$$

where $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$. The Fisher information matrix has determinant

$|I_n(\lambda,\mu)| \propto (\mu^3\lambda)^{-1}$, and hence, Jeffreys' vague prior (Box and Tiao [2]) is $p(\mu,\lambda) \propto (\mu^3\lambda)^{-1/2}$, which when combined with the likelihood (2.1) does not produce a tractable or proper posterior distribution. Also, following the vague prior idea of Box and Tiao [2] and taking $p(\mu|\lambda) \propto$ constant and $p(\lambda) \propto \lambda^{-1}$, mathematically intractable posterior distributions for estimating $R(t|\mu,\lambda)$ are obtained. It is assumed here that μ, the mean life, is known, and Jeffreys' noninformative prior $p(\lambda) \propto \lambda^{-1}$ is used for λ. In addition, the gamma family of distributions is a natural conjugate family for λ, and Bayes estimators of $R(t|\mu,\lambda)$ for this case will be indicated.

For the improper prior $p(\lambda) \propto \lambda^{-1}$, the posterior distribution of λ, given $\underline{x}$, is from (2.1)

$$p(\lambda|\underline{x},\mu) = K\,\lambda^{-1} \exp\left[-\frac{\lambda}{2\mu^2}\sum_{i=1}^{n}(x_i-\mu)^2/x_i\right], \tag{2.2}$$

where the constant K is given by

$$K = \Gamma(\tfrac{n}{2})\left[\frac{1}{2\mu^2}\sum_{i=1}^{n}(x_i-\mu)^2/x_i\right].$$

Hence, $p(\lambda|\underline{x},\mu)$ is a gamma distribution of the form

$$p(\lambda|\underline{x},\mu) = [\Gamma(\alpha)\beta^{\alpha}]^{-1}\,\lambda^{\alpha-1}\exp(-\lambda/\beta),\ \lambda > 0,$$

with $\alpha = n/2$ and $\beta = 2\mu^2/\sum_{i=1}^{n}[(x_i-\mu)^2/x_i]$. Then with respect to a squared-error loss function, the Bayes estimator of λ is $\hat{\lambda}_B = n\mu^2/\sum_{i=1}^{n}[(x_i-\mu)^2/x_i]$, which is the same as the mle of λ when μ is known. For $R(t|\mu,\lambda)$, with respect to squared-error loss, the Bayes estimator for the improper prior is $\hat{R}_B(t) = E_{\Lambda}[R(t|\mu,\Lambda)|\underline{x}]$. Thus, from (1.2), for each $t > 0$,

$$\hat{R}_B(t) = E_\Lambda[\,\Phi((\Lambda/t)^{1/2}\,(1-t/\mu))\,]$$

$$- E_\Lambda[\exp(2\Lambda/\mu)\quad \Phi(-(\Lambda/t)^{1/2}\,(1+t/\mu))\,]. \tag{2.3}$$

To evaluate the first expected value in (2.3), Lemma 1 of Padgett and Wei [11] may be applied with c in that lemma equal to $t^{-1/2}(1 - t/\mu)$. Thus,

$$E_\Lambda[\,\Phi\,((\Lambda/t)^{1/2}\,(1+t/\mu))\,] = P[T_{2\alpha} < (1-t/\mu)(\alpha\beta/t)^{1/2}], \tag{2.4}$$

where α and β are parameters of the posterior pdf $p(\lambda|\underline{x},\mu)$ defined previously and T_ν denotes a random variable having Student's t-distribution with ν degrees of freedom. The second expected value on the right-hand side of (2.3) is evaluated similarly after absorbing the exponential term into the posterior density (2.2). Again, by Lemma 1 of [11] with $c = -t^{-1/2}(1+t/\mu)$,

$$E_\Lambda[\exp(2\Lambda/\mu)\quad \Phi(\,-(\Lambda/t)^{1/2}\,(1+t/\mu))]$$

$$= (1-2\beta/\mu)^{-\alpha}\,P\{T_{2\alpha} < -\,(1+t/\mu)[\alpha\beta\mu/(t(\mu-2\beta))]^{1/2}\}. \tag{2.5}$$

Therefore, the Bayes estimator $\hat{R}_B(t)$ is given by (2.5) subtracted from (2.4). This estimate may be easily computed since it involves only probabilities for the t distribution.

If the gamma family of priors with parameters γ and δ in the form $p(\lambda) \propto \lambda^{\gamma-1}\exp(-\lambda/\delta)$ is used for λ, then the same kind of expected values are obtained as in (2.3). Applying Lemma 1 of [11] again yields the Bayes estimator of reliability as

$$\hat{R}_G(t) = P[T_{2\gamma*} < c_1(\gamma*/\delta*)^{1/2}]$$

$$- [\delta*/(\delta*-2/\mu)]^{\gamma*}\,P\{T_{2\gamma*} < c_2[\gamma*/(\delta*-2/\mu)]\},$$

where $\gamma* = \gamma + n/2$,

$$\delta^* = \delta^{-1} + (2\mu^2)^{-1} \sum_{i=1}^{n} (x_i-\mu)^2/x_i, \quad c_1 = t^{-\frac{1}{2}}(1-t/\mu), \text{ and } c_2 = -t^{-\frac{1}{2}}(1+t/\mu).$$

These results bear a resemblance to those of the log-normal (or normal) failure model obtained in [11] (see also [1]). Also, it should be remarked that for the noninformative prior $p(\mu|\lambda) \propto$ constant, $p(\lambda) \propto \lambda^{-1}$, estimates of $R(t|\mu,\lambda)$ may be obtained by numerical integration, but a closed-form expression for the estimator seems extremely difficult to obtain. If both λ and μ are unknown, one may be tempted to use the mℓe, $\bar{x} = \hat{\mu}$, in the expressions (2.4) and (2.5) to obtain an estimate of reliability $\tilde{R}_B(t)$. The effect of this is indicated in the next section by some computer simulation results.

3. MONTE CARLO SIMULATIONS

Since direct comparisons of the behavior of various estimators of $R(t|\mu,\lambda)$ are not feasible due to the mathematical complexity of the estimators, Monte Carlo simulations were performed. The maximum likelihood (ML) and minimum variance unbiased (MVU) estimators when μ is known were compared with the corresponding Bayes estimator (2.3). For several values of t, μ, and λ, 2000 samples of size n (= 10, 20, 30) were generated and the average squared errors (ASE) and average estimated reliability (AER) were computed for each estimator. Similar to the results in [11], the Bayes estimator had an overall smaller mean squared error than the ML and MVU estimators, as anticipated. For the case that μ and λ both were unknown, the estimator $\tilde{R}_B(t)$ suggested at the end of Section 2 (using (2.3) with μ replaced by $\bar{x} = \hat{\mu}$) was compared with the ML and MVU estimators given in [6] in the same kind of simulation procedure. Surprisingly, this estimator performed as well as the MVU estimator in the sense of average value and did not have a uniformly larger ASE than either the ML or MVU estimator. Some of the results of the simulations in the latter case are given in Table 1.

Table 1. Average Estimated Reliability and Average Squared Error ($\times 10^{-4}$)
Based on 2000 Samples of Size n (μ and λ Unknown)

			n = 10						n = 30					
			$\tilde{R}_B(t)$		MVUE		MLE		$\tilde{R}_B(t)$		MVUE		MLE	
(μ,λ)	t	R(t)	AER	ASE	AER	ASE	AER	ASE	AER	ASE	AER	ASE	AER	ASE
(1,.25)	1	.238	.240	244	.239	104	.214	104	.239	95	.238	29	.229	30
	3	.077	.088	39	.094	48	.076	24	.078	12	.078	15	.071	12
(3,.25)	1	.332	.299	587	.332	112	.308	133	.326	338	.331	27	.323	30
	3	.167	.176	288	.175	74	.140	59	.170	126	.166	21	.152	21
	5	.115	.134	179	.132	59	.099	32	.118	64	.116	17	.102	15
(1,1)	1	.332	.326	192	.324	130	.310	144	.336	76	.334	46	.329	48
	3	.047	.056	14	.055	32	.052	20	.048	3	.047	9	.046	8
(3,1)	1	.570	.537	538	.572	151	.577	177	.561	239	.572	43	.574	46
	3	.258	.261	238	.259	108	.235	111	.261	91	.259	33	.250	34
	5	.156	.163	119	.161	84	.138	67	.156	38	.155	26	.146	25
	7	.105	.113	61	.114	63	.096	40	.105	17	.105	18	.098	16
(5,1)	1	.616	.551	694	.617	140	.627	161	.586	338	.614	42	.618	44
	5	.223	.226	251	.225	96	.197	93	.224	99	.223	29	.218	29
	10	.120	.131	106	.130	66	.105	42	.121	34	.120	20	.110	18
(3,4)	1	.845	.770	262	.830	69	.842	60	.836	80	.856	29	.860	27
	3	.349	.328	170	.327	133	.313	150	.351	70	.349	47	.345	49
	5	.153	.153	58	.152	85	.138	72	.152	18	.152	27	.146	26
	7	.072	.080	22	.079	46	.073	32	.073	6	.072	14	.070	12
(5,4)	1	.904	.828	289	.902	55	.909	44	.878	83	.905	18	.907	17
	5	.318	.312	191	.313	123	.296	136	.316	74	.315	41	.309	43
	10	.120	.126	54	.128	74	.114	57	.120	14	.119	22	.114	20

4. AN EXAMPLE

As an example, the estimator $\tilde{R}_B(t)$ as well as the ML and MVU estimators were used to estimate reliability for several values of t from the n=46 repair time observations (in hours) for an airborne communication transceiver ([17] and [6]). Chhikara and Folks [6] obtained a good fit to this data by the inverse Gaussian distribution with $\hat{\mu} = \bar{x} = 3.61$ and $\hat{\lambda} = 1.704$. The estimates of reliability are given in Table 2.

Table 2. Estimates of Reliability

t	1	2	3	5	10	15
$\tilde{R}_B(t)$	0.6934	0.4578	0.3305	0.1984	0.0789	0.0388
MLE	0.6986	0.4607	0.3325	0.1996	0.0791	0.0386
MVUE	0.6951	0.4618	0.3368	0.2057	0.0829	0.0396

5. CONCLUSION

For the case that the mean lifetime μ in the inverse Gaussian model is known, the posterior distribution of λ is easily obtained for the Jeffreys prior and the natural conjugate prior as indicated by Banerjee and Bhattacharyya [1]. For this case the Bayes estimators of reliability given in Section 2 resemble the analogous results in the log-normal (or normal) model. If both μ and λ are unknown, the Bayes solution for reliability in a compact form seems to be extremely difficult, at least for the parametric form (1.1). It also seems to be even more difficult to obtain a Bayes estimator for the failure rate function or mean residual life. Hence, an estimator, $\tilde{R}_B(t)$, of reliability was proposed for this case in Section 2, and its properties were indicated as a result of computer simulations. For other Bayesian inferences on reliability, numerical integrations must be performed in any actual application to obtain the posterior distribution of reliability.

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